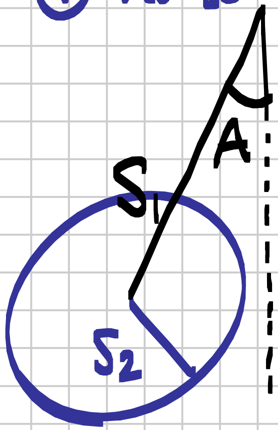
C2 Exercise 2C (two solutions to sine rule)

Note Title 23/01/2007

This exercise assumes you have an understanding of the conditions of congruence of triangles, and especially the condition ASS in which there is an ambiguity since there are four cases for the set ASS:

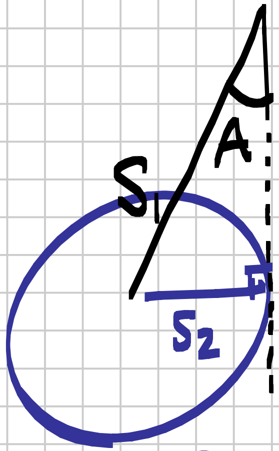
① no solution




$S_2 < S_1 \sin A$

or $S_1 = S_2$

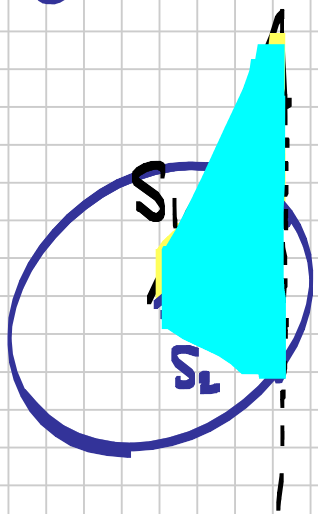
② one solution




$\sin A = \dfrac{S_2}{S_1}$

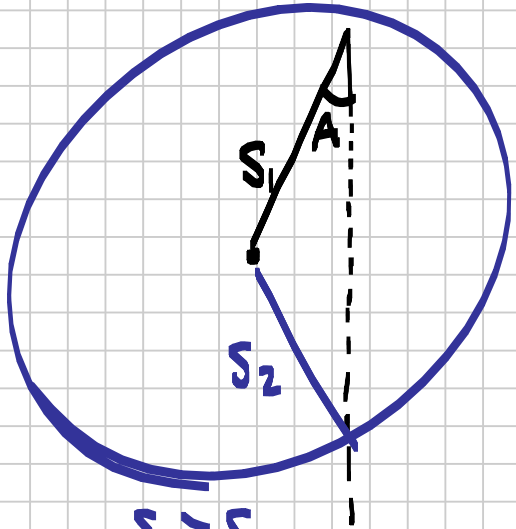
$\Rightarrow S_2 = S_1 \sin A$

③ two solutions



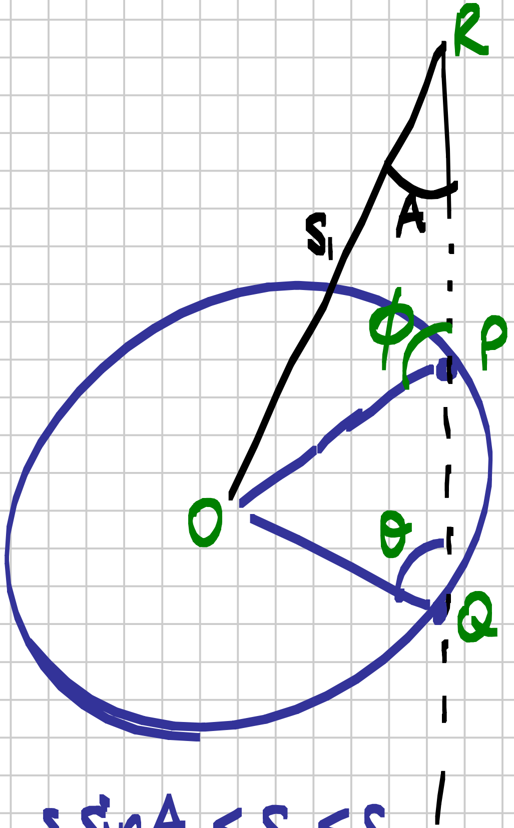

$S_1 \sin A < S_2 < S_1$

④ one solution




$S_2 > S_1$

Now if we look more carefully at case 3:




$s_1 \sin A < s_2 < s_1$

Consider the solution triangles OPR and OQR.

Why should the answers for $\theta$ and $\phi$ be such that
$\phi = 180 - \theta$?

Well $\triangle OPQ$ is isosceles so $Q\hat{P}O = \theta$ and $\phi + O\hat{P}Q = 180$ since on a straight line

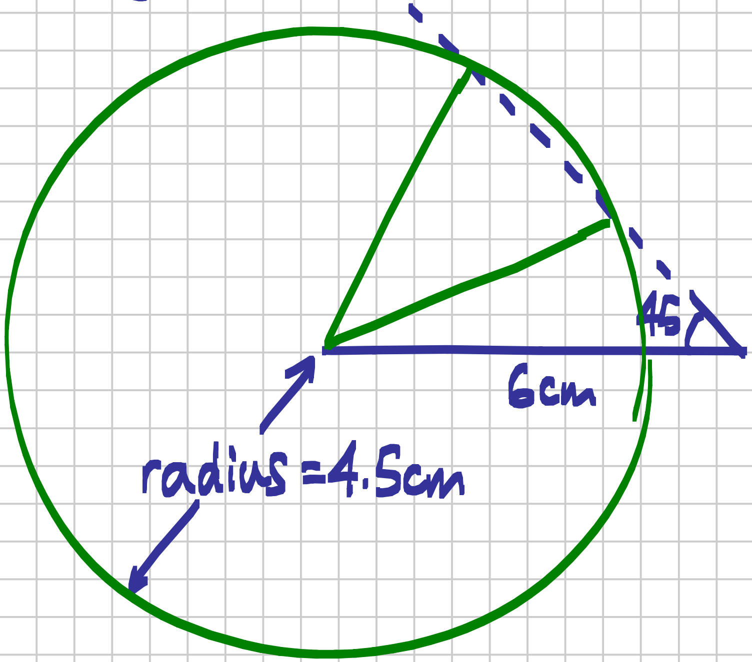
So $\phi = 180 - \theta$.

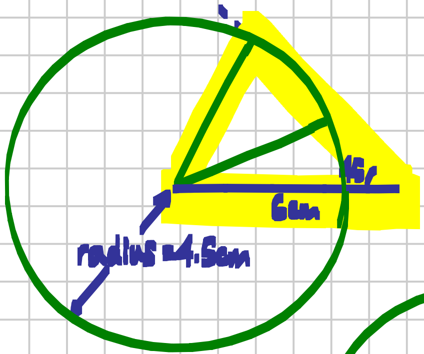
note that $A < \theta < 90°$ is equivalent to the condition $s_1 \sin A < s_2 < s_1$ for case 3.

1 In $\triangle ABC$, $BC = 6cm$, $AC = 4.5cm$ and $\angle ABC = 45°$

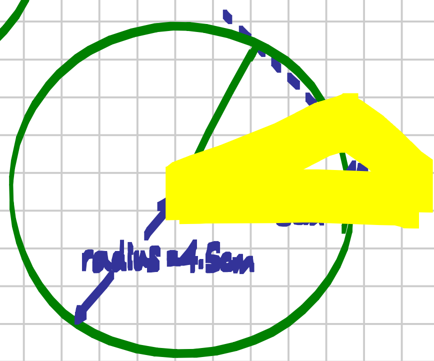
a Calculate the two possible values for $\angle BAC$.

We have the blue data in the question. Construct the green solutions:



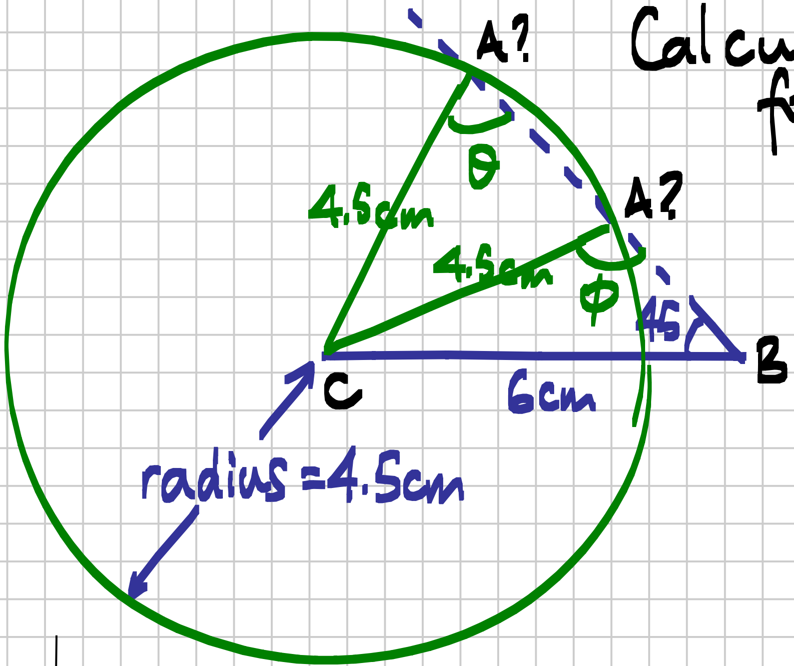




or...



C2 Ex 2C

1 a
b




Calculate the two possible values for $\angle BAC$ Call them $\theta$ & $\phi$.

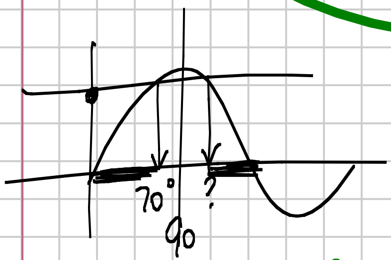
Use sine rule to find $\theta$:

$$\frac{\sin\theta}{6} = \frac{\sin 45}{4.5}$$

$$\Rightarrow \sin\theta = \frac{6}{4.5}\sin 45^\circ \; *$$

$$\Rightarrow \theta = \sin^{-1}\left(4/3 \sin 45^\circ\right)$$

$$\Rightarrow \theta = 70.528779 37^\circ \simeq 70.5^\circ$$




Now from our proof on page 2 $\phi = \underline{180 - \theta} \simeq 109.5^\circ$

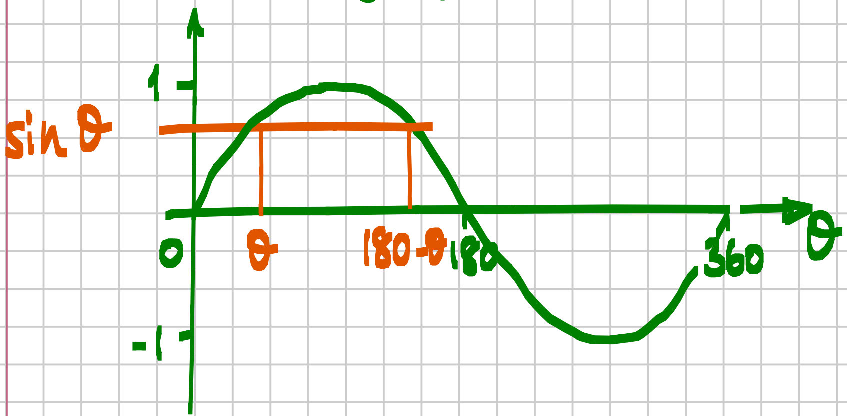
C2 Ex 2C

1 Plenary... * You might at this stage wonder if the sine rule works since there are <u>two</u> solutions.

Well, return to the rule $\frac{\sin\theta}{6} = \frac{\sin 45}{4.5}$

$$\Rightarrow \sin\theta = \frac{6}{4.5}\sin 45^\circ \quad *$$

Note the graph of $\sin\theta$ for $0 < \theta < 360^\circ$




and realise than since

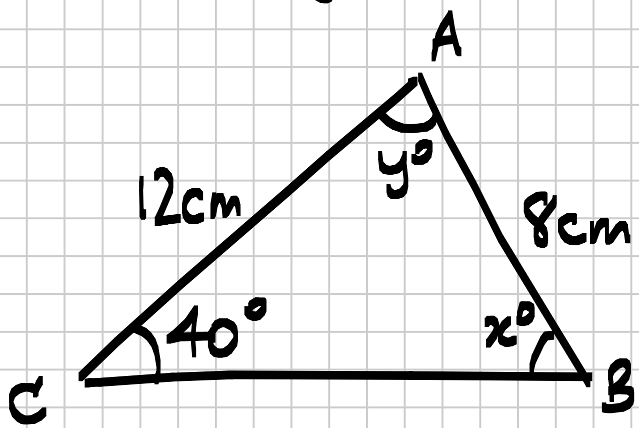
$$\sin(180-\theta) = \sin\theta \quad \text{for } 0<\theta<90$$

we can rewrite * as

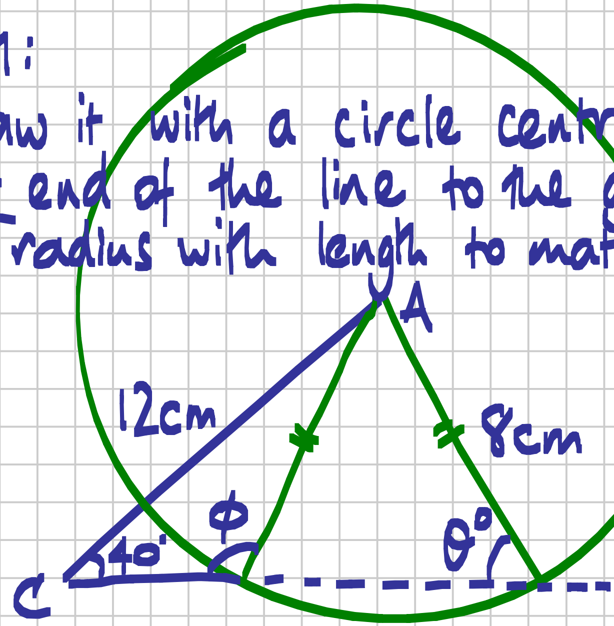
$$\sin(180-\theta) = \frac{6}{4.5}\sin 45^\circ$$

C2 Ex 2C

2 a) ... calculate the possible values of $x$ and the corresponding values of $y$.




Step 1:
redraw it with a circle centred on the other end of the line to the given angle with radius with length to match 2nd side




Step 2: you can now see why $x$ might vary. Use the sine rule to find $\theta$. Then $\phi = 180 - \theta$.
The values $\theta$ & $\phi$ are your two possible values for $x$

$$\frac{\sin\theta}{12} = \frac{\sin 40}{8} \Rightarrow \theta = \sin^{-1}\left(\frac{12}{8}\sin 40\right) = 74.61856831$$

C2 Ex 2C

2a) ctd... so $x \simeq 74.6°$ or $x \simeq 105°$

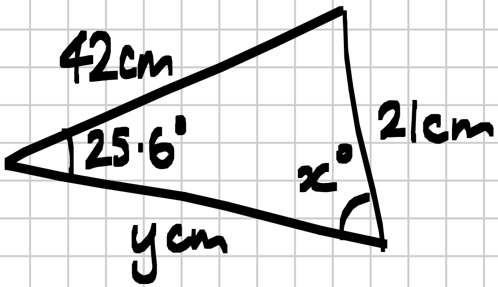
When $x \simeq 74.6°$, $y = 180 - (40 + \theta) \simeq 65.4°$

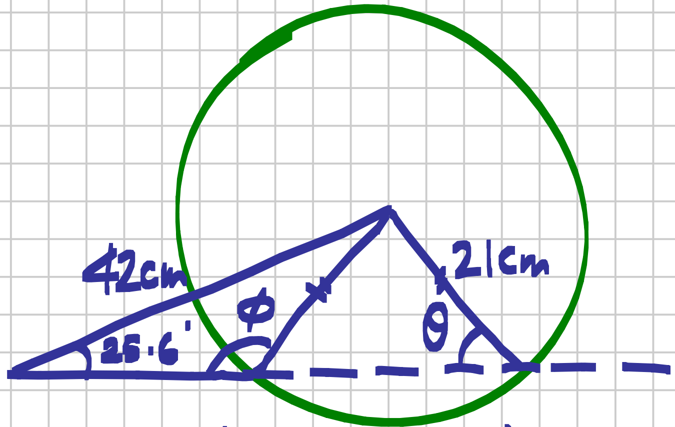
When $x \simeq 105°$, $y = 180 - (40 + \phi) \simeq 34.6°$

Be careful to pair up values of $x$ and $y$ like you might in a simultaneous-equations-with-quadratics question.

C2 Ex 2C

2 b) ... calculate the possible values of $x$ and the corresponding values of $y$.



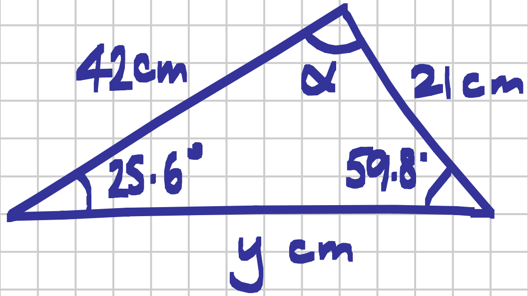




$$\frac{\sin\theta}{42} = \frac{\sin 25.6}{21} \Rightarrow \theta = \sin^{-1}\left(\frac{42}{21}\sin 25.6\right) = 59.78823534$$

$\phi = 180 - \theta = 120.2117647$ so $x \simeq 59.8°$ or $120°$

C2 Ex 2C

2 b) ctd... When $x \simeq 59.8°$

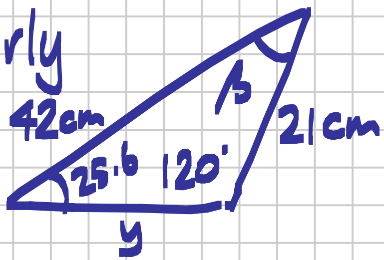



$$\alpha = 180 - (25.6 + \theta) = 94.61176466°$$

$$\frac{y}{\sin\alpha} = \frac{21}{\sin 25.6} \Rightarrow y = \frac{21 \sin\alpha}{\sin 25.6} = 48.444115$$

[redacted]

Similarly



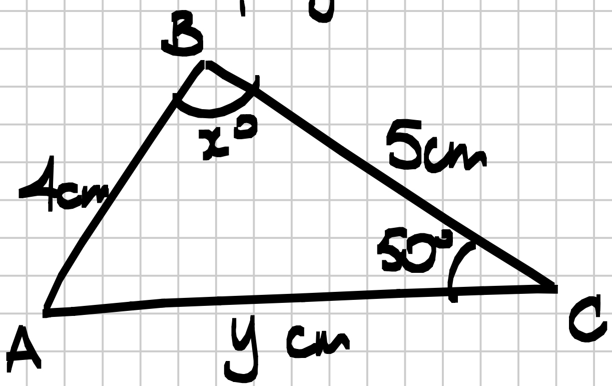

$$\beta = 180 - (25.6 + \phi) = 34.18823534°$$

$$y = \frac{21 \sin\beta}{\sin 25.6} = 27.30982072$$

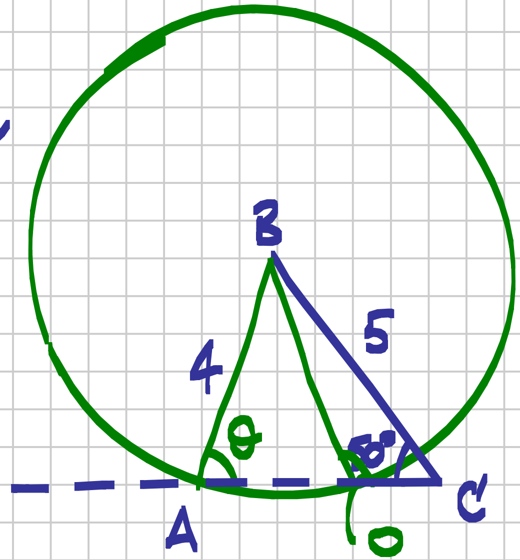
[redacted]

C2 Ex 2C

2 c) ... calculate the possible values of $x$ and the corresponding values of $y$.




Add the circle and $\theta$:




$$\frac{\sin\theta}{5} = \frac{\sin 50}{4} \Rightarrow \theta = \sin^{-1}\left(\frac{5}{4}\sin 50°\right) = 73.24685774 \simeq 73.2°$$

$$x = 180 - (50 + \theta) \simeq 56.8° \quad \text{or} \quad \phi = 180 - \theta \Rightarrow x = \theta - 50 = 23.2°$$

$$\frac{y}{\sin x} = \frac{4}{\sin 50°} \Rightarrow y = \frac{4\sin x}{\sin 50} = 4.366933182 \simeq 4.37\text{cm}$$

C2 Ex 2C

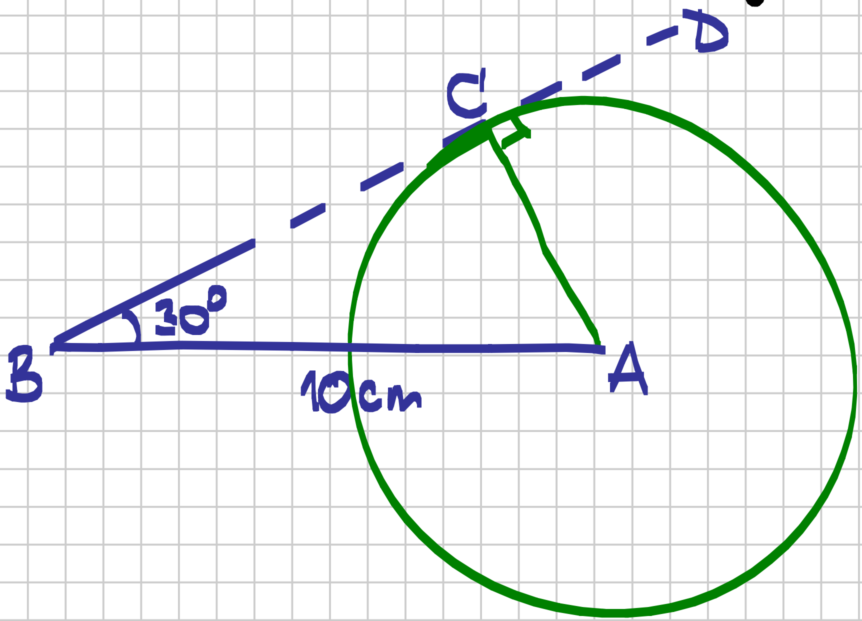
2c ctd When $x \simeq 56.8°$ then $y \simeq 4.37$ cm

When $x \simeq 23.2°$ $y = \dfrac{4\sin x}{\sin 80°} = 2.060942914 \simeq 2.06$ cm

C2Ex2C

3 In each of the following cases $\triangle ABC$ has $\angle ABC = 30°$ and $AB = 10$ cm.

a Calculate the least possible length that AC could be.




The smallest circle that intersects BD is when $A\hat{C}B = 90°$.
This results in the least possible length AC.

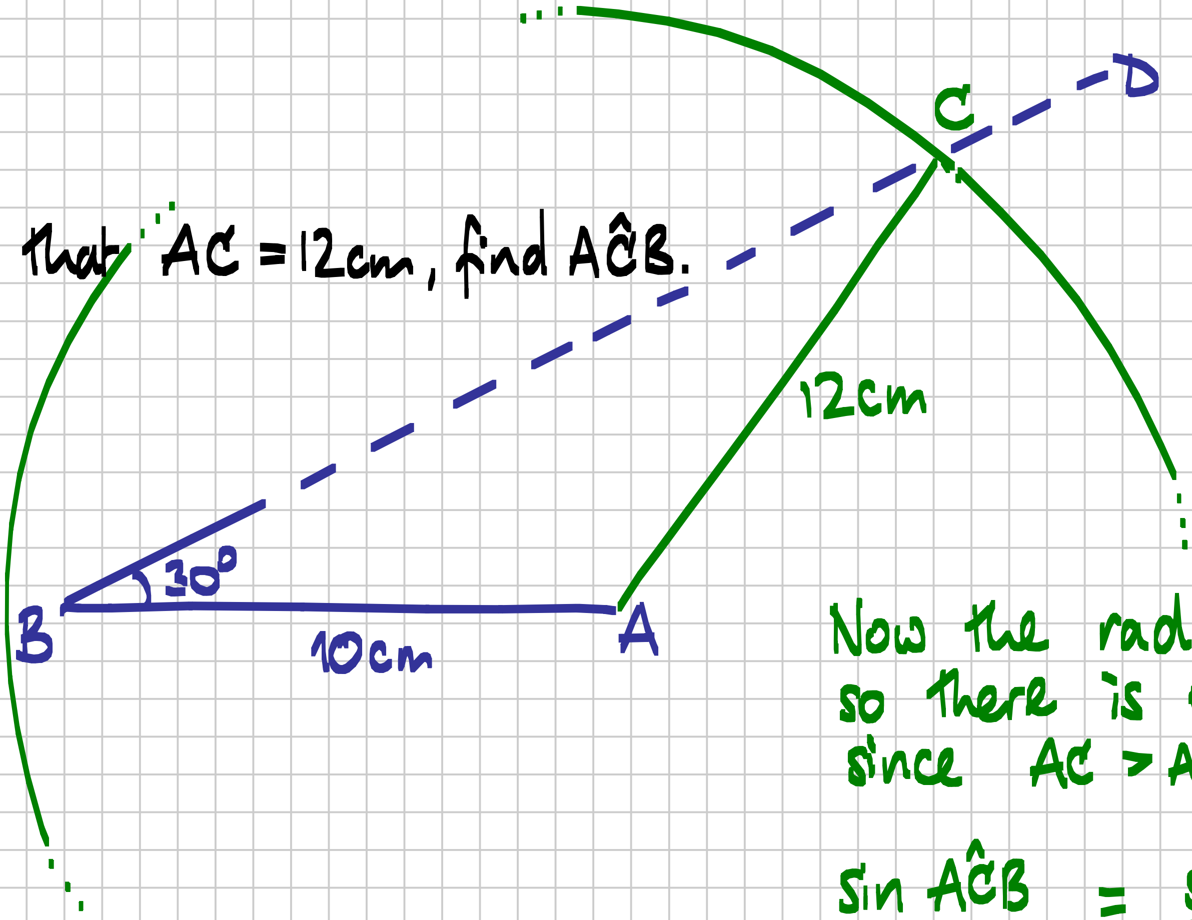
$$\sin 30 = \frac{AC}{10}$$

$$\Rightarrow AC = 10 \sin 30°$$

$$\Rightarrow AC = 5 \text{ cm}$$

C2 Ex 2C

3b Given that AC = 12cm, find $A\hat{C}B$.




Now the radius AC = 12 cm so there is only one intersection since AC > AB.

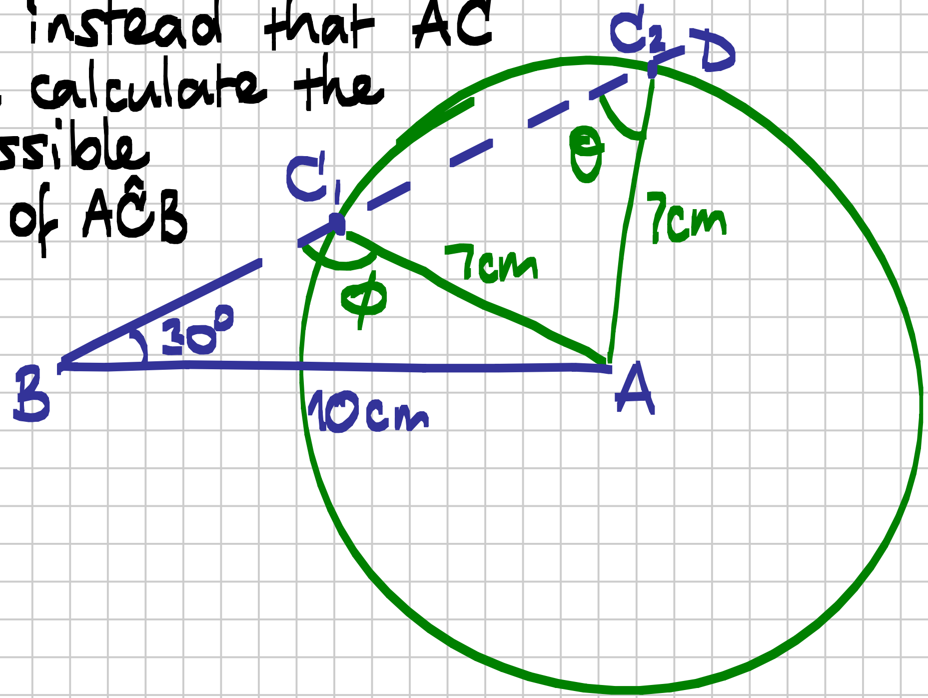
$$\frac{\sin A\hat{C}B}{10} = \frac{\sin 30}{12}$$

$$\Rightarrow \quad A\hat{C}B = \sin^{-1}\left(\frac{10}{12}\sin 30^\circ\right)$$

so $ACB = 24.62431835 \simeq 24.6^\circ$

C2 Ex 2C

3c Given instead that AC = 7cm, calculate the two possible values of $A\hat{C}B$




$$\frac{\sin\theta}{10} = \frac{\sin 30}{7}$$

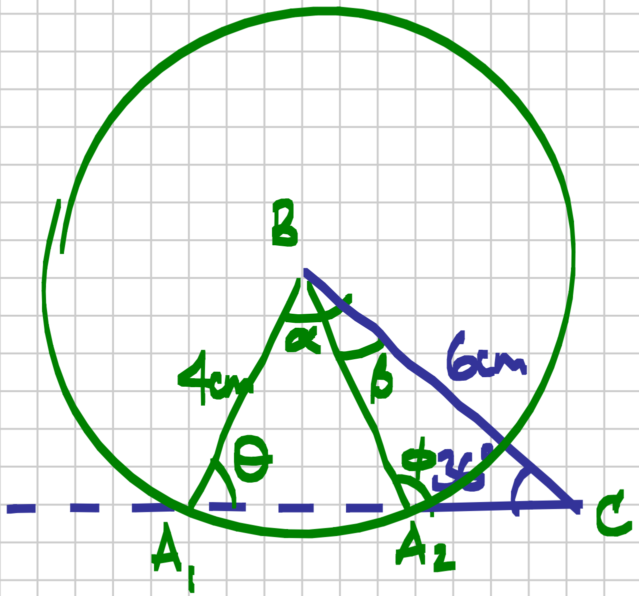
$$\theta = \sin^{-1}\left(\frac{10}{7}\sin 30\right)$$

$$\theta = 45.5846914°$$
$$\theta \approx 45.6°$$

$$\phi = 180 - \theta \approx 134°$$

C2 Ex 2C

4 Triangle $A\hat{B}C$ is such that $AB = 4$ cm, $BC = 6$ cm and $\angle ACB = 36°$.
Show that one of the possible values of $\angle ABC = 25.8°$ (to 3 s.f.).
Using this value, calculate the length of AC.




Call $B\hat{A}_1C = \theta$ , $B\hat{A}_2C = \phi$

Similarly $C\hat{B}A_1 = \alpha$ , $C\hat{B}A_2 = \beta$

Aim: show $\alpha$ or $\beta = 25.8°$

$$\frac{\sin\theta}{6} = \frac{\sin 36}{4} \Rightarrow \theta = \sin^{-1}\left(\frac{6}{4}\sin 36\right)$$

$$\Rightarrow \theta = 61.845432° \Rightarrow \theta \simeq 61.8°$$

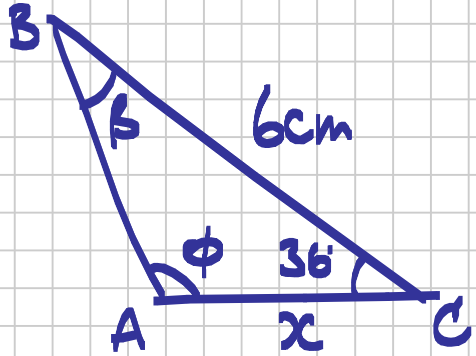
$$\phi = 180 - \theta \Rightarrow \phi \simeq 118°$$

$$\alpha = 180 - (36 + \theta) \simeq 82.1° \quad \text{or} \quad \beta = 180 - (36 + \phi) = 25.845432°$$

C2 Ex 2C

4 ctd So $\beta \approx 25.8°$ as required.

Given this triangle find length $AC = x$




$$\frac{6}{\sin\phi} = \frac{x}{\sin\beta}$$

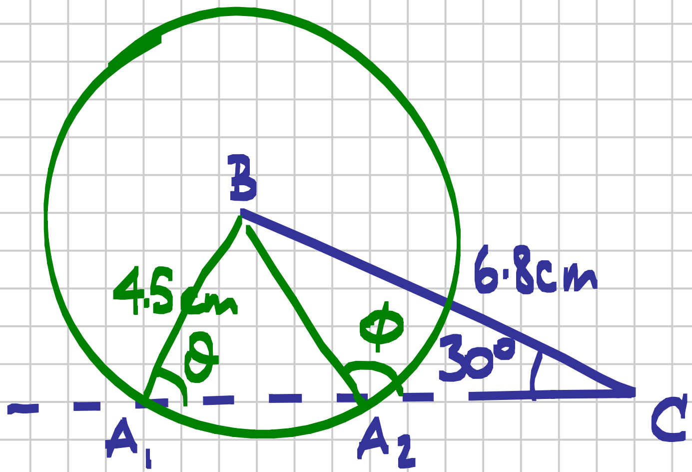
$$\Rightarrow x = \frac{6\sin\beta}{\sin\phi}$$

$x = 2.96669477$

$x \approx 2.97$ cm

C2 Ex 2C

5 Two triangles ABC are such that AB = 4.5 cm, BC = 6.8 cm and $\angle ACB = 30°$. Work out the value of the largest angle in each of the triangles.




$$\frac{\sin\theta}{6.8} = \frac{\sin 30}{4.5}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{6.8}{4.5}\sin 30\right)$$

$$\theta = 49.0739366\ldots \simeq 49.1°$$

$$\phi \simeq 131°$$

In the triangle $A_1BC$ the largest angle is $180-(30+\theta)$

So it is $\simeq 101°$, in the triangle $A_2BC$ the largest angle is $131°$.